Equilibrium spin currents and non-Abelian diamagnetism in condensed matter

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Content of the talk

1. Spin-orbit interaction and the “problem” of definition of spin currents.

2. Local SU(2) gauge invariance, conservation laws, and spin currents.

3. Implications for equilibrium state: Non-Abelian Landau diamagnetism.

4. Gauge *co*variance *vs.* gauge *in*variance and the “field-strength copies”

5. Concluding remarks: How serious is the “problem of definition”?

Equation of motion for the spin density

I. “First principle” many-body Hamiltonian with spin-orbit interaction

$$H=\sum_{\alpha=1}^{N}\left[\frac{\mathbf{P}_\alpha^2}{2m}+e\varphi\left(\mathbf{r}_\alpha,t\right)\right]+\frac{1}{2}\sum_{\alpha\neq\beta}V\left(\mathbf{r}_\alpha-\mathbf{r}_\beta\right)+H_{SO}$$

$$H_{SO}=\sum_{\alpha=1}^{N}\left[\frac{e\hbar}{8m^2c^2}\left\{\mathbf{P}_\alpha,\mathbf{E}\left(\mathbf{r}_\alpha\right)\times\boldsymbol{\sigma}_\alpha\right\}+\frac{e\hbar^2}{8m^2c^2}\mathrm{div}\mathbf{E}\left(\mathbf{r}_\alpha\right)\right]$$

Nontrivial correction –
spin-orbit interaction

Darwin term – can be absorbed
into the scalar potential

$$H_{SO}\sim\sum_{\alpha=1}^{N}\mathbf{B}_{\mathrm{eff}}\left(\mathbf{P}_\alpha,\mathbf{r}_\alpha\right)\times\boldsymbol{\sigma}_\alpha \longrightarrow$$

Generates precession in an effective
nonlocal “magnetic field”

$$\left[H_{SO},\mathbf{S}\right]\neq 0$$

II. "Intuitive" definitions of the spin density and the spin current

$$s^b(\mathbf{r}) = \sum_{\alpha=1}^{N} \sigma_\alpha^b \delta(\mathbf{r}-\mathbf{r}_\alpha); \qquad J_i^b(\mathbf{r}) = \frac{1}{2}\sum_{\alpha=1}^{N}\left\{v_{\alpha,i}, \sigma_\alpha^b \delta(\mathbf{r}-\mathbf{r}_\alpha)\right\}$$

where $\mathbf{v}_\alpha = \dfrac{\partial H}{\partial \mathbf{P}_\alpha} = \dfrac{1}{m}\left(\mathbf{P}_\alpha + \dfrac{e\hbar}{2mc^2}\mathbf{E}(\mathbf{r}_\alpha)\times\boldsymbol{\sigma}_\alpha\right)$ is the velocity operator

Spin continuity equation: $\partial_t s^a(\mathbf{r}) = i\left[H, s^a(\mathbf{r})\right]$

$$\partial_t s^a(\mathbf{r},t) + \partial_i J_i^a(\mathbf{r},t) = \mathscr{T}^a(\mathbf{r},t)$$

effective spin torque induced by SO interaction

Non-uniqueness of the spin current: A redefinition of the spin current can be compensated by correcting a torque to keep the continuity equation unchanged

For the "intuitive" definition

$$\mathscr{T}^a = \frac{e\hbar}{2mc^2}\delta_k^a\left(E_i J_k^i - E_k J_i^i\right)$$

Is the problem of the definition of spin current unique in physics?

1. Flow of particles: $$\partial_t n + \nabla \mathbf{j} = 0, \quad \mathbf{j} \to \mathbf{j} + \nabla \times \mathbf{f}$$
2. Energy-momentum tensor: $$\partial_\mu \theta^{\mu\nu} = 0, \qquad \theta^{\mu\nu} \to \theta^{\mu\nu} + \partial_\lambda f^{\lambda\mu\nu}$$
3. Stress tensor: $$m\partial_t j_k + \partial_i \Pi_{ik} + n\partial_k U = 0, \qquad \Pi_{ik} \to \Pi_{ik} + \partial_j f_{jik}$$

The "currents" become physically unique when the coupling to the corresponding gauge fields is introduced into the theory

1. Electro-magnetic field – Maxwell equations $$\partial_\mu F^{\mu\nu} = 4\pi e j^\nu$$
2. Gravitational field – Einstein equations $$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G \theta_{\mu\nu}$$

"Continuity equations" with unique currents follow from a certain local invariance of the theory

$$j^\mu = \frac{\delta H}{\delta A_\mu}; \qquad \theta^{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta L}{\delta g_{\mu\nu}}$$

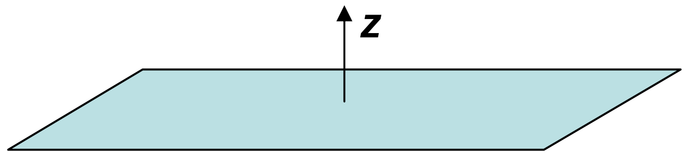
The variational, symmetry based definition of currents is possible without physical presence of real gauge fields $\longrightarrow$ $$\Pi_{ij} = -\frac{2}{\sqrt{g}} \frac{\delta H}{\delta g^{ij}}$$

Equilibrium spin currents in a "Rashba medium"

(E. I. Rashba, Phys. Rev. B **68**, 241315(R) (2003) [1])

Quasi-2d electrons in a semiconductor (e.g. GaAs) quantum well

$$H = \frac{\mathbf{p}^2}{2m} - \alpha\left(p_x\sigma^y - p_y\sigma^x\right) - \beta\left(p_x\sigma^x - p_y\sigma^y\right)$$



Rashba term due to structure inversion ($z \to -z$) asymmetry

Dresselhaus term (bulk inversion asymmetry)

By the direct calculation for non-interacting electron gas Rashba has found equilibrium (ground state) homogeneous spin currents flowing in this system

$$J_x^y = -J_y^x = \frac{m}{6\pi^2}\alpha^3, \qquad \alpha \neq 0, \quad \beta = 0$$ "Rashba medium"

$$J_y^y = -J_x^x = \frac{m}{6\pi^2}\beta^3, \qquad \beta \neq 0, \quad \alpha = 0$$ "Dresseshaus medium"

What is the physical significance (if there is any) of these strange currents?

They are exactly equivalent to diamagnetic currents responsible for the Landau diamagnetism!

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[1] E. B. Sonin, Phys. Rev. B **76**, 033306 (2007); Phys. Rev. Lett. **99**, 266602 (2007)

Local SU(2) gauge invariance of spin-orbit Hamiltonians

$$H = \int d\mathbf{r} \left\{ \frac{1}{2m} \left[\left(i\partial_i + \mathcal{A}_i \right) \Psi \right]^\dagger \left[\left(i\partial_i + \mathcal{A}_i \right) \Psi \right] - \Psi^\dagger \mathcal{A}_0 \Psi + U \Psi^\dagger \Psi \right\} + H_{\text{int}}$$

where $\mathcal{A}_\mu = \mathcal{A}_\mu^a \tau^a$, $\tau^a = \frac{1}{2}\sigma^a$ - generators of SU(2) (spin-½ operators)

$$\boxed{\begin{array}{ll} \mathcal{A}_0 = -\dfrac{e\hbar}{mc} B^a \tau^a, \quad \mathcal{A}_i = \dfrac{e\hbar}{mc^2} \varepsilon_{ija} E_j \tau^a & \text{Pauli Hamiltonian} \\ \mathcal{A}_x = 2m\left(\beta\tau^x - \alpha\tau^y\right), \quad \mathcal{A}_y = 2m\left(\beta\tau^x - \alpha\tau^y\right), \quad \mathcal{A}_z = 0 & \text{Rashba-Dresselhaus Hamiltonian} \end{array}}$$

The corresponding spin-orbit action $S\left[\Psi, \mathcal{A}_\mu\right] = \int dt \left(i\Psi^\dagger \partial_t \Psi - H \right)$

is invariant with respect to local non-Abelian gauge transformations

$$\Psi \to \mathcal{U}\Psi, \qquad \mathcal{A}_\mu \to \mathcal{U}\mathcal{A}_\mu\mathcal{U}^{-1} - i\left(\partial_\mu \mathcal{U}\right)\mathcal{U}^{-1}$$

$$\mathcal{U}(\mathbf{r}, t) = \exp\left[i\theta^a(\mathbf{r}, t) \tau^a \right]$$

Non-Abelian gauge invariance implies a ***covariant*** conservation of the "color" current

$$\boldsymbol{J}_\mu = J_\mu^a \tau^a = \frac{\delta S}{\delta \boldsymbol{\mathcal{A}}_\mu}$$

$$D_t \boldsymbol{J}_0 + D_i \boldsymbol{J}_i = 0, \qquad D_\mu \bullet = \partial_\mu \bullet - i\left[\boldsymbol{\mathcal{A}}_\mu, \bullet\right]$$

Explicitly the currents and the covariant continuity equation take the form:

$$J_0^a = \frac{\delta S}{\delta \mathcal{A}_0^a} = \Psi^\dagger \tau^a \Psi = s^a\left(\mathbf{r}, t\right) \qquad \text{- Spin density}$$

$$J_i^a = \frac{\delta S}{\delta \mathcal{A}_i^a} = \frac{-i}{2m}\left[\Psi^\dagger \tau^a \partial_i \Psi - (\partial_i \Psi^\dagger)\tau^a \Psi\right] - \frac{\mathcal{A}_i^a}{2m}\Psi^\dagger \Psi \qquad \text{- Spin current}$$

$$\overbrace{\partial_t J_0^a + \varepsilon^{abc} \mathcal{A}_0^b J_0^c}^{\text{Covariant time derivative}} + \overbrace{\partial_i J_i^a + \varepsilon^{abc} \mathcal{A}_i^b J_i^c}^{\text{Covariant space derivative}} = 0$$

Precession in the usual magnetic field

Internal spin-orbit torque

Under the gauge transformation $\boldsymbol{J}_\mu \to \mathcal{U} \boldsymbol{J}_\mu \mathcal{U}^{-1}; \quad D\boldsymbol{J}_\mu \to \mathcal{U} D\boldsymbol{J}_\mu \mathcal{U}^{-1}$

transform covariantly

Non-Abelian Landau diamagnetism: Equilibrium spin currents

The current is given by the derivative of the energy $E\left[\mathcal{A}_i^a\right]=\langle H\rangle$
which should depend on gauge invariant combinations of the vector potential

The only local gauge covariant object is the field strength

$$\boldsymbol{\mathcal{F}}_{ij}=\partial_i\boldsymbol{\mathcal{A}}_j-\partial_j\boldsymbol{\mathcal{A}}_i-i\left[\boldsymbol{\mathcal{A}}_i,\boldsymbol{\mathcal{A}}_j\right]$$

An example: Homogeneous semiconductor with Rashba-Dresselhaus SO coupling

$$E_{SO}=\frac{\lambda}{4}\int \mathrm{tr}\left[\boldsymbol{\mathcal{F}}_{ij}\boldsymbol{\mathcal{F}}_{ij}\right]d\mathbf{r}=\frac{\lambda}{4}\int \mathcal{F}_{ij}^{\,a}\mathcal{F}_{ij}^{\,a}\,d\mathbf{r},\qquad J_i^a=\frac{\delta E_{SO}}{\delta\mathcal{A}_i^a}$$

$$\boldsymbol{J}_j=\lambda D_i\boldsymbol{\mathcal{F}}_{ij}=\lambda\left(\partial_i\boldsymbol{\mathcal{F}}_{ij}-i\left[\boldsymbol{\mathcal{A}}_i,\boldsymbol{\mathcal{F}}_{ij}\right]\right)$$

We got exactly the Yang-Mills magnetostatics equation!

In the case of space-independent spin-orbit coupling parameters

$$\boldsymbol{J}_j = -i\lambda\left[\boldsymbol{\mathcal{A}}_i, \boldsymbol{\mathcal{F}}_{ij}\right] = -\lambda\left[\boldsymbol{\mathcal{A}}_i,\left[\boldsymbol{\mathcal{A}}_i, \boldsymbol{\mathcal{A}}_j\right]\right] \sim \alpha^3$$

Confirmation by microscopic calculations for non-interacting system

$$J_i^a = \sum_{\omega}\sum_{\mathbf{p}} \mathrm{tr}\left[\frac{p_i}{m}\tau^a G(\omega,\mathbf{p})\right] - \frac{n}{4m}\mathcal{A}_i^a; \qquad G^{-1}(\omega,\mathbf{p}) = i\omega + \mu - \frac{1}{2m}\left(p_i - \mathcal{A}_i^a\tau^a\right)^2$$

The result of calculations for the ground state (zero temperature)

$$J_i^a = \frac{N_F}{24m^2}\left(\mathcal{A}_j^a\mathcal{A}_i^b\mathcal{A}_j^b - \mathcal{A}_i^a\mathcal{A}_j^b\mathcal{A}_j^b\right) \equiv -\frac{N_F}{24m^2}\left[\boldsymbol{\mathcal{A}}_i,\left[\boldsymbol{\mathcal{A}}_i, \boldsymbol{\mathcal{A}}_j\right]\right]$$

$$\boxed{J_x^y = -J_y^x = \frac{m}{6\pi^2}\alpha\left(\alpha^2 - \beta^2\right), \qquad J_x^x = -J_y^y = \frac{m}{6\pi^2}\beta\left(\alpha^2 - \beta^2\right)}$$

$\boldsymbol{\mathcal{F}}_{ij} = 4m^2\left(\alpha^2 - \beta^2\right)\tau^z$ - Rashba-Dresselhaus color magnetic field

Direct connection to the Landau diamagnetism

Linear response to the non-Abelian vector potential (SO interaction)

$$J_i^a = \int d\mathbf{r}\chi_{ij}^{ab}(\mathbf{r},\mathbf{r}')\mathcal{A}_j^b(\mathbf{r}')$$

$$\chi_{ij}^{ab}(\mathbf{r},\mathbf{r}') = \left\langle\left\langle J_i^a(\mathbf{r}); J_j^b(\mathbf{r}')\right\rangle\right\rangle - \delta(\mathbf{r}-\mathbf{r}')\delta_{ij}\delta^{ab}\frac{n}{4m}$$

In the long-wavelength limit for non-interacting system we get the Landau response function!

$$\chi_{ij}^{ab}(\mathbf{q}) = \frac{N_F}{24m^2}\delta_{ij}\left(q_i q_j - q^2\delta_{ij}\right)$$

$$\boxed{J_j^a = \frac{N_F}{24m^2}\partial_i\left(\partial_i\mathcal{A}_j^a - \partial_j\mathcal{A}_i^a\right) \to \frac{N_F}{24m^2}D_i\mathcal{F}_{ij}^a}$$

The connection to the Landau diamagnetic coefficient still holds for a general interacting case if the interaction is spin-independent!

Field-strength copies in condensed matter

We have found that the energy is expressed in terms of the simple invariant:

$$E_{SO} \sim \int \mathrm{tr}\left[\mathcal{F}_{ij}\mathcal{F}_{ij}\right]d\mathbf{r}, \qquad \mathcal{F}_{ij} = \partial_i \mathcal{A}_j - \partial_j \mathcal{A}_i - i\left[\mathcal{A}_i, \mathcal{A}_j\right]$$

However the same field strength can correspond to different potentials, for example

$$\left.\begin{array}{l} \mathcal{A}_x = 2m\left(\beta\tau^x - \alpha\tau^y\right), \quad \mathcal{A}_y = 2m\left(\beta\tau^x - \alpha\tau^y\right) \\ \mathcal{A}_x = -2m^2\left(\alpha^2 - \beta^2\right)y\tau^z, \quad \mathcal{A}_y = 2m^2\left(\alpha^2 - \beta^2\right)x\tau^z \end{array}\right\} \longrightarrow \mathcal{F}_{ij} = 4m^2\left(\alpha^2 - \beta^2\right)\tau^z$$

This phenomena is known as a “Wu-Yang ambiguity” (Phys.Rev.D **12,** 3843 (1975))

In our spin-orbit context it leads to identical energy of two different distributions of spin currents!

Non-local invariants: Wilson loop integral

$$W_C = P\exp\left\{i\oint \mathcal{A}_i dx^i\right\} \approx 1 + i\mathcal{F}_{ij}\Delta S^{ij}$$

Only for infinitesimal loop determined by the strength

Beyond the lowest order in the field or when non-local effects are important
the energies corresponding to different copies become different!

Concluding remarks

In most real system SO coupling corresponds to non-Abelian fields with nonzero magnetic component. This component inevitably generates diamagnetic spin currents, which appear as one of the most universal phenomena in condensed matter.

Unfortunately these currents do not directly couple to the common experimental probes, which make their detection extremely difficult!

On the formal side the gauge-invariant formulation of the SO problem

- **Clarifies the question of the definition of the spin currents**
- **Allows to efficiently control microscopic calculations and phenomenological construction of effective theories of spin dynamics in the presence of SO**
- **Leads to a clear and physically intuitive interpretation of formal results**
- **Makes a number of nice connections of condensed matter physics to High-energy physics, which could probably be useful for both parties**