Nonequilibrium Spin Dynamics: From Protons in Water to a
Gauge Theory of Spin-Orbit Coupling

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Nonequilibrium dynamics of spin degrees of freedom in condensed matter, ranging from classical liquids to solids and ultracold atomic gases, is one the focus topics in physics. Here we present a gauge theory of spin dynamics in spin-orbit coupled gases for a “pure” gauge realization of the spin-orbit coupling field. This approach allows one to describe the spin dynamics of Fermi gases in terms of exact general response functions and to map it on the density dynamics in a dual system without spin-orbit coupling. We apply this approach to electrons in disordered two-dimensional structures and to cold atomic gases of interacting fermions with synthetic spin-orbit coupling at very low temperatures.

1. Introduction.

Nonequilibrium spin dynamics, often seen as a simple exponential decay of initial magnetization with time, became a topic of a great interest in the studies of magnetic resonance in liquids and solids about seven decades ago. In liquids the macroscopic magnetization decays due to random in time magnetic interactions between the nuclei, caused by their chaotic motion. A theory, based on the conventional density matrix approach, was developed and applied to the understanding of the linewidth of the nuclear magnetic resonances. One of the most important features in this understanding is the so called “motional narrowing”, that is slowing down the spin relaxation and the corresponding narrowing of the resonance line in the presence of quick and random variations of interactions between the nuclei\(^1\).

In the last decades, spin-related phenomena in semiconductors became a rapidly developing research field\(^2,3\). Here the injected (e. g. optically) spin density relaxes due a relativistic coupling of the electron spin to its...
translational motion, usually referred to as a “spin-orbit coupling”. In typical bulk semiconductors, like GaAs, the spin-orbit coupling is proportional to the cube of the electron momentum. Usually bulk semiconductor samples are rather dirty because the conduction electrons there are produced by ionized donors. As a result, the spin precession induced by spin-orbit coupling is very slow compared to the momentum relaxation. Here, the application of the Boltzmann kinetic equation gives the famous Dyakonov-Perel’ relaxation mechanism, corresponding to the expected in this regime motional narrowing. With the fast development of semiconductor technologies, two-dimensional electron systems with the linear in the momentum spin-orbit coupling in the Rashba and Dresselhaus forms became available. For such systems, with still relatively low mobilities, the Dyakonov-Perel’ mechanism also corresponds to the leading channel of the spin relaxation. Interesting dependences of spin dynamics on the relative strength of the Rashba and Dresselhaus terms were predicted and observed in these two-dimensional electron gases. Recently, novel high-quality semiconductor structures, where spin precession time caused by the spin-orbit coupling is of the order of or even less than the momentum relaxation time, and, therefore, the spin dynamics is not in the motion narrowing regime anymore, have been produced and studied.

In the last few years, very unusual systems with spin-orbit coupling came to the light. It was shown that artificial spin-orbit coupling arising from the Doppler effect, can be optically produced in ensembles of both bosonic and fermionic cold atoms. A detailed study of spin dynamics in ultracold trapped atomic gases is becoming experimentally feasible now. In particular, two systems attracted a great deal of attention. One of them is the spin-orbit coupled Bose-Einstein condensates with the pseudospin 1/2 degree of freedom. The other class is represented by the cold fermion isotopes and studied in Ref. 15. In both cases, in addition to the SOC, an effective Zeeman magnetic field can be produced optically. Relative strength of SOC compared to the kinetic energy is much larger than that achievable in semiconductors. This opens a venue to the observation of new effects, which can not be seen in semiconductor spintronics structures.

Our interest here is in the fact that this linear in momentum coupling can be generally, independent on the system dimension, described in terms of non-Abelian and, at certain conditions, Abelian gauge field. In this gauge theory approach, the analysis of the spin dynamics can be done in a general and elegant form, which allows to mutually map spin and the particle density evolutions both for irreversible and reversible processes.
and, as a result, study spin dynamics in terms of generalized diffusion in the absence of spin-orbit coupling.

Here we present a theory of spin dynamics in the presence of a special "pure gauge" spin-orbit field, that is the field which can be completely gauged away by a proper transformation. With the inverse transformation to the initial gauge, we recover in general terms the full spin dynamics. We show that several regimes of the macroscopic spin motion can easily be explored using a single formula based on the spin susceptibility of the system without spin-orbit coupling.

2. Motional narrowing and spin relaxation: from liquids to semiconductors and from semiconductors to cold atoms.

The problem of spin relaxation is condensed matter was for the first time formulated for protons in liquids. It has been recognized that the active chaotic motion of molecules makes the relaxation time of the magnetization longer and, as a result, the width of the absorption line becomes smaller. In a liquid, the position of a molecule changes in time showing a diffusive random walk behavior. The magnetic moments of protons interact via a magnetic dipole-dipole interaction and thus produce a magnetic field that rapidly changes in time because of the random molecular jumps. The precession of spins in this fluctuating field is the physical origin of the magnetization relaxation. It turns out that the spins precess relatively slowly on the time scale of the molecular motion, which eventually leads to the effect of the motional narrowing. Indeed, the magnetic interaction between protons is of the order of $\mu_n^2/V_p$, where $\mu_n = e\hbar/2m_pc$ with $m_p$ being the proton mass and $V_p$ being the volume per proton, respectively. The corresponding precession rate $\Omega_\mu$ is of the order of $\mu_n^2/\hbar V_p$. The characteristic time for changing the dipole configuration due to the molecular motion is $\tau_{dc} \ll 1/\Omega_\mu$ and the spin precession angle $\delta \phi$ is of the order of $\tau_{dc}^2 \Omega_\mu \ll 1$.

In the long time limit $t \gg \tau_{dc}$ one obtains a typical diffusive behavior $\langle \phi^2 \rangle \sim (\tau_{dc} \Omega_\mu)^2 t/\tau_{dc}$, which translates to the spin relaxation rate $\Gamma_s$ of the order of $\tau_{dc} \Omega_\mu^2$. As a result, the magnetization can be almost frozen in liquid water compared to a fast relaxation in ice.

The problem of spin relaxation, now for electrons, reappeared in the research focus about two decades later, with the studies of optical spin injection in bulk semiconductors of the III-V group, such as, for example, GaAs. The Hamiltonian of the spin-orbit coupling in this materials has the
Dresselhaus form, which is cubic in the electron momentum:

\[ H_{so} = \alpha_0 (\kappa \cdot \sigma), \]

where \( \kappa_x = k_x (k_y^2 - k_z^2) \) and other components are obtained by cyclic permutation of Cartesian indices, and \( \sigma^a \) with \( a = \{x, y, z\} \) are the Pauli matrices. Surprisingly enough, the exact value of the Dresselhaus coupling constant \( \alpha_0 \) is not yet known, although it is well established that it is of the order of 10 eV \( \cdot \overset{\alpha^3}{\text{\AA}} \). It is convenient to describe the spin evolution by solving the kinetic equation for the momentum dependent 2 \( \times \) 2 density matrix \( f(k) \)

\[ \frac{\partial f}{\partial t} = -\frac{i}{\hbar} [H_{so}, f] + St \{ f \}, \]

where the first term in the right hand side describes the spin precession due to SOC, and the collision integral \( St \{ f \} \) is responsible for the momentum relaxation (randomization of the electron motion on the characteristic time scale \( \tau \)). The total spin is then calculated as follows

\[ S^a = \text{tr} \int f(k)\sigma^a \frac{d^3k}{(2\pi)^3}. \]

At the beginning of 70s, semiconductors were dirty and the corresponding momentum relaxation time \( \tau \) was small. In addition, the experiments were usually done at relatively high temperatures, when phonons further enhance the momentum relaxation rate. Solution of the Boltzmann equation in the limit \( \alpha_0 k^3 \tau / \hbar \ll 1 \) (i. e. in the collision dominated regime) naturally shows the motional narrowing with \( \Gamma_s \) of the order of \( \tau \left( \frac{\alpha_0}{\hbar} \right)^2 \langle k^6 \rangle \), where
⟨...⟩ stands for the statistical average. This result is valid for any temperature and mechanism of the momentum relaxation, either due to disorder or due to phonons.26

The progress in semiconductor technologies allowed researchers to produce heterostructures, which can host two-dimensional electron gases. In such systems the spin-orbit coupling is usually represented as a sum of the Rashba (due to the asymmetry of heterostructure)27 and Dresselhaus (due to the lack of microscopic inversion symmetry of the material unit cell)28 terms. The common form of such a coupling is exemplified by the Rashba-Dresselhaus Hamiltonian

\[ H_{so} = \alpha_R (k_x \sigma_y - k_y \sigma_x) + \alpha_D (k_x \sigma_x - k_y \sigma_y), \]

which is typical for an asymmetric quantum well grown along [001] crystallographic axis. We note that the most general linear in momentum SOC for 2D systems can be written as follows

\[ H_{so} = \alpha_x k_x (h_x \cdot \sigma) + \alpha_y k_y (h_y \cdot \sigma), \]

where \( \alpha_x \) and \( \alpha_y \) are the coupling constants, and \( h_x \) and \( h_y \) are the corresponding unit vectors. In semiconductors, typical values of \( \alpha \) vary from \( 10^{-12} \) eV·cm for Si-based to \( 10^{-8} \) eV·cm for InSb-based structures.

In the collision dominated regime, application of the motion narrowing approach to this SOC naturally yields the Dyakonov-Perel spin relaxation. However, an interesting new effect occurs here if the system is characterized by one rather than by two directions of the SOC field. Since the component of spin parallel to this common axis commutes with the SOC Hamiltonian, it remains constant while the orthogonal components decay exponentially in the conventional way. As a result, spin relaxation becomes strongly anisotropic. This can be reached in the balanced Rashba-Dresselhaus system7,8 with \( \alpha_D = \pm \alpha_R \), which corresponds to collinear \( h_x \) and \( h_y \). Another example is (110) GaAs quantum wells28-31 where \( h_x = (0, 0, 1) \), and \( \alpha_y = 0 \) with the well axes chosen with respect to the crystal as \( x \parallel [1\bar{1}0], y \parallel [001], \) and \( z \parallel [110] \). In both systems the observed spin relaxation rate is indeed strongly anisotropic with the life time for one spin component being a few orders of magnitude larger than for the others. The relaxation rate for these components is determined by the mechanisms different from the homogeneous spin-orbit coupling.31,32

The above spin-orbit coupling can be characterized by a characteristic length scale \( L_{so} = \hbar^2 / m_\alpha \). The significance of this length, which will play an important role in our analysis, can be understood as follows. For an
electron moving with the velocity \( v = p/m \) the spin precession rate \( \Omega_{\text{so}} \) is of the order of \( \alpha m v \), and, therefore, the corresponding precession angle is \( \alpha m L \), where \( L = vt \) is the electron displacement. Hence, \( L_{\text{so}} \) is the displacement at which the electron spin undergoes a substantial rotation. Typical values of \( L_{\text{so}} \) can be of interest. For GaAs with \( m = 0.067 m_0 \), where \( m_0 \) is the free electron mass, and \( \alpha/\hbar \) is of the order of \( 10^6 \) cm/s, \( L_{\text{so}} \) is of the order of 2 micron. For cold atomic gases such as \( ^{40} \text{K} \), we have \( m \sim 10^5 m_0 \) and \( \alpha/\hbar \sim 1 \) cm/s. It is interesting to observe that the product \( \alpha m \) is not changed much from semiconductors to cold atoms, and the spin precession length is of the order of 1 micron for both systems.

Nowadays, clean systems, where the motional narrowing regime is not anymore valid, can be produced. One example is given by quantum wells with high electron mobilities, where electron free path is of the order of or larger than \( L_{\text{so}} \). Even a better example is provided by cold atomic Fermi gases. For a unified description of these systems one needs a general theory of the macroscopic spin dynamics, which we will present below.

3. Non-Abelian field and gauge transformation

The main observation underlying our approach to the spin dynamics is the following. Let us introduce a \( 2 \times 2 \) matrix valued vector

\[
A_j = A_j^a \sigma^a = m \alpha_j (h_j \sigma), \tag{6}
\]

where \( A_j^a = 2m \alpha_j h_j^a \). Using this vector we can compactly represent the most general form of the linear in momentum SOC

\[
H_{\text{so}} = \frac{1}{2m} \{ A_j, p_j \}. \tag{7}
\]

Accordingly, the full Hamiltonian for many (in general interacting) fermions in the presence of such SOC can be written as follows:

\[
\hat{H} = \int d^{D} r \left\{ \Psi^\dagger \left( i \nabla_j + \frac{A_j^a \sigma^a}{2m} + B^a \sigma^a \right) \Psi \right\} + \hat{H}_{\text{int}} [\Psi^\dagger, \Psi], \tag{8}
\]

where \( D = 2, 3 \) is the dimensionality of the system, \( \Psi \) is a spinor fermionic field operator, \( \mathbf{B} \) is the external Zeeman field, and the term \( \hat{H}_{\text{int}} [\Psi^\dagger, \Psi] \) describes spin-independent inter-particle interactions and the external potential, random for disordered semiconductor or regular for the trapping potential for cold fermions (here and below we use \( \hbar \equiv 1 \)).

The Hamiltonian (8) shows that the SOC and the Zeeman field enter the problem as an effective background non-Abelian SU(2) field. The spatial
components $A_j$ of the SU(2) four-potential correspond to SOC, while its
time component $B^b$ describes the Zeeman field. The convenience of this
representation, which is valid an arbitrary nonuniform spin-orbit coupling,
is related to the form invariance of the Hamiltonian with respect to a general
local SU(2) gauge transformation. Indeed, let us perform a local, at given
$r$-point, SU(2)-rotation in the spin subspace by
\[ U = e^{i\theta^a(r)\sigma^a/2} \]
with the transformation of the field operators,
\[ \tilde{\Psi}^+ U^{-1} = \Psi^+, \quad \tilde{\Psi} = U^{-1} \Psi, \]
and the SU(2) potentials
\[ \tilde{A}_i = U^{-1} (i\partial_i U) + U^{-1} A_i U, \quad \tilde{B}^b \sigma^b = U^{-1} B^b \sigma^b U. \]
After inserting the transformed operators and the potentials into the Hamil-
tonian (8) we observe that the latter preserves its form. This exact sym-
metry can be very helpful for the theory of spin dynamics in the presence
of SOC.

It is worth noting that the above SU(2) gauge transformation keeps
the spin-independent quantities such as the charge and the current densi-
ties invariant (this in particular implies the SU(2) invariance of interaction
term $\hat{H}_{\text{int}}[\Psi^\dagger, \Psi]$ in the full Hamiltonian). However, the spin-dependent
quantities, such as the spin density operators, $S = S^a \sigma^a$ transforming as
\[ S = U \tilde{S} U^{-1} \]
correspond to covariant observables. The difference between the invariant
and covariant observables under the gauge transformation is crucial for the
understanding of the spin dynamics. For the matrix $U = \exp \left( i\theta (h \cdot \sigma)/2 \right)$, where $h$ is a unit length vector, transformed $\sigma^b$–matrices acquire the form:
\[ \tilde{\sigma}^b = \cos \theta \sigma^b + \sin \theta \varepsilon^{abc} h^a \sigma^c + 2 \sin^2 \frac{\theta}{2} h^b (h \sigma), \]
where $\varepsilon^{abc}$ is the Levi-Civita tensor, and the product $(h \cdot \sigma)$ remains con-
stant under this transformation. Therefore, if we present $S$ as the sum of
longitudinal and transverse components $S = S_{\parallel} + S_{\perp}$ with $S_{\parallel} = h (S \cdot h)$,
the longitudinal component (spin projection at the $h$–axis remains con-
stant), and the $S_{\perp}$ changes. This simple observation will be important for
the discussion in this paper.

Let us now assume that the SU(2) potential $A_i$ is a pure gauge, that
is both $A_x$ and $A_y$ can be removed by the above transformation such that
$\tilde{A}_x = \tilde{A}_y = 0$. In this case there exists a local rotation determined by three coordinate-dependent functions $\theta^A(x,y)$

$$U_A = e^{i\sigma_A(r)\sigma^a/2}$$

(13)

such that both initial components $A_i$ can be represented in the form

$$A_i = U_A (i\partial_i U_A^{-1}).$$

(14)

With this choice

$$\tilde{A}_i = U_A^{-1} (i\partial_i + U_A (i\partial_i U_A^{-1})) U_A$$

(15)

vanishes. If the spin-orbit field can be removed by a gauge transformation, the subsequent spin dynamics is fully described with the invariant observables. Then, the inverse SU(2)-rotation restores the physical values of the spin components. We present and follow this program in this paper.

We mention by passing an example using the same idea of transformation in classical electrodynamics. When the motion of a relativistic charge in static perpendicular electric field $E$ and magnetic field $H$ is considered, there exists a reference frame, where, after the Lorentz transformation, the smaller of these fields is zero. In this frame the equations of the electron motion become simple, and in the case $H < E$, where the magnetic field vanishes, are, essentially, one-dimensional. The inverse Lorentz transformation provides the full description of the electron motion in the total static electromagnetic field.

Vector-potential is a pure gauge, allowing to remove six terms in $A_x, A_y$, with the transformation $U_A$ based on three functions $\theta_A^a(\rho)$ only at certain relations between the $A_x$ and $A_y$ components. The corresponding conditions are formulated with the field tensor for non-Abelian field $F_{ij}$:

$$F_{ij} = \partial_i A_j - \partial_j A_i - i [A_i, A_j] = 0.$$

(16)

Taking into account the commutation relation

$$[h, \sigma, h, \sigma] = \frac{i}{2} \sigma [h, \sigma \times h, \sigma],$$

(17)

we obtain

$$A_j = 2m\alpha \nu_j (h, \sigma).$$

(18)

where $h = h_i = h_j$ if $\alpha_i, \alpha_j \neq 0$ or $h = h_j$ for nonzero $\alpha_i$, where $a = x$ or $a = y$, $\alpha = (\alpha_x^2 + \alpha_y^2)^{1/2}$, and $\nu$ is a unit vector. As a result, the Hamiltonian of spin-orbit coupling acquires the form $H_{so} = \alpha (k \cdot \nu) (h \cdot \sigma)$. The
corresponding gauge transformation:

$$U_A = \exp \left[ 2im_\alpha \sigma_j (h\sigma) \right] \exp \left[ 2im_\alpha \nu_i (h\sigma) \right] = \exp \left[ 2im_\alpha (h\sigma) \cdot (r \cdot \nu) \right].$$

(19)

As we have already mentioned the transformation (19) leaves invariant all spin-independent quantities, while the spin density transforms covariantly:

$$\tilde{S} = \frac{1}{2} \text{tr} \left\{ \sigma U_A^{-1} (S\sigma) U_A \right\}. \quad (20)$$

Since the spin-orbit coupling is gauged away the dynamics of the transformed spin density $\tilde{S}(r,t)$ reduces to the spin dynamics in the electron gas without SO interaction, which significantly simplifies calculations. Then, the physical spin density $S(r,t)$ can be restored as follows

$$S = \frac{1}{2} \text{tr} \left\{ \sigma U_A (\tilde{S}\sigma) U_A^{-1} \right\}, \quad (21)$$

to obtain the measurable results. Here we follow this program and show that this approach allows to describe all regimes of spin dynamics on the same footing.

After the local SU(2) transformation $U_A = e^{i\theta (r) \tau^a}$, the Hamiltonian of Eq. (8) reads

$$\hat{H} = \int d^3r \left\{ \tilde{\Psi}^\dagger \left[ \frac{-\nabla^2}{2m} + \sigma \cdot \tilde{B}(r) \right] \tilde{\Psi} \right\} + \hat{H}_\text{int} [\tilde{\Psi}^\dagger, \tilde{\Psi}], \quad (22)$$

where $\tilde{B}(r) = \text{tr} \{ \sigma U_A^{-1}(r) (\sigma B) U_A(r) \}/2$ is the transformed Zeeman field.

When a uniform spin density is produced, then from Eq. (20) we find that it is mapped to the following spin texture

$$\tilde{S}(r,0) = h \quad (23)$$

where the parallel to the vector $h$ term is untouched by the gauge transformation. The orthogonal to $h$ part of the spin transforms to the helix structure$^{34,35}$$^3$

$$\tilde{S}_\perp (r,0) = \left[ S - h (S \cdot h) \right] \cos (Q_h \cdot \rho) - (S \times h) \sin (Q_h \cdot \rho) \quad (24)$$

with $Q_h = 2ma \nu$ being the helix wave vector.

4. General expressions with susceptibilities

The spin relaxation is probed by applying the Zeeman-like field of the form $B(t) = B\theta(-t)$. That is, we bring the system to an equilibrium
uniform polarized state by a static Zeeman coupling and then switch it off at \( t = 0 \). After the polarizing field is released, the spin density relaxes to zero by the combined action of the SOC-induced spin precession, disorder, and interatomic collisions. To describe this process we first eliminate the SO by the gauge transformation of Eq. (19) and produce helix structure in Eq. (24). In the following evolution, the uniform part of the initial spin distribution remains constant. The perpendicular component (24) is described by generalized diffusion equation:

\[
\tilde{S}_\alpha^\perp (r, t) = \int D_{\alpha\beta}^\perp (r - r', t) \tilde{S}_\beta^\perp (r', 0) d^D r',
\]

where \( D_{\alpha\beta}^\perp (r, t) \) is the exact spin diffusion Green’s function including the effects of disorder and interaction between the particles. In a nonmagnetic system without SO coupling the spin diffusion Green’s function is diagonal in spin indexes \( D_{\alpha\beta}^\perp (r, t) = \delta_{\alpha\beta} D(r, t) \), and only the Fourier components of \( D(r, t) \) with the modulus of the wave vector \( q = Q_h \) contribute to the helix evolution, Eq. (24). Hence Eq. (25) simplifies as follows

\[
\tilde{S}_\perp (r, t) = \tilde{S}_\perp (r, 0) D(Q_h, t),
\]

where \( D(q, t) \) is a Fourier component of the spin diffusion Green’s function

\[
D(q, t) = \int d^D r e^{-i q r} D(r, t).
\]

Since the time-dependent factor in Eq. (26) is scalar the transformation back to the physical spin, Eq. (21), is trivial. Hence by simply removing “tildes” in Eq. (26) we get for the observable spin evolution

\[
S_\perp (t) = S_\perp (0) \int \frac{d\omega}{2\pi} D(Q_h, \omega) e^{-i\omega t}.
\]

In Eq. (28) we represented \( D(q, t) \) via the Fourier integral because in the \( \omega \)-domain there is a simple expression of the spin diffusion Green’s function \( D(q, \omega) \) in terms of the Fourier component of the spin-spin correlation function (the spin response function) \( \chi_{\sigma\sigma}(Q_h, \omega) \)

\[
D(Q_h, \omega) = \frac{1}{i\omega} \left[ \frac{\chi_{\sigma\sigma}(Q_h, \omega)}{\chi_{\sigma\sigma}(Q_h, 0)} - 1 \right].
\]

This equation results from linear response to a time-dependent magnetic field that is adiabatically switched on at \( t = -\infty \), and then suddenly switched off at \( t = 0 \), i.e. \( B(t) = e^{\delta t} \theta(-t) B \) (see e.g. similar calculations in Ref. 52). This expression is the exact solution in terms of the spin susceptibility of a nonmagnetic system at any given experimental conditions.
The SOC enters the spin dynamics solely via the wave vector dependence of the spin-spin correlation function $\chi_{\sigma\sigma}(q,\omega)$ of the SOC-free system. Hence by monitoring the spin evolution for different values of $q$ one can access the entire momentum and frequency dependence of $\chi_{\sigma\sigma}(q,\omega)$. The spin evolution in the physical system corresponds to the “washing out” of a spatially inhomogeneous spin texture in a dual system without SO coupling. The physical spin relaxation of $S_\perp(t)$, which is always parallel to its initial direction, occurs due to the spin precession arising from SO coupling, and randomness introduced by the disorder and interactions between particles. In the transformed picture this is related to the diagonal structure of the spin response in a nonmagnetic gas. For the real system this translates to the fact that spins of particles with opposite momenta precess around the $\hbar$-axis in the opposite directions with the same rate, that is of the time reversal symmetry.

In semiconductors SO coupling energy is much smaller than the kinetic energy of electrons. As a result $Q_h \ll \langle k \rangle$, where $\langle k \rangle$ is the mean electron momentum. Therefore in electron gases we replace the static response function in the denominator in Eq. (29) by the macroscopic finite-temperature Pauli spin susceptibility, $\chi_{\sigma\sigma}(Q_h, 0) \approx \chi_P$. At the level of the random phase approximation the spin response function $\chi_{\sigma\sigma}(q,\omega)$ is equal to the density response function $\chi(q,\omega)$ and the Pauli susceptibility $\chi_P$ coincides with compressibility $\partial n/\partial \mu$, that is with the density of states at the Fermi level at low temperatures. Hence the spin diffusion Green’s function entering Eq. (28) reduces to

$$D(Q_h,\omega) = \frac{1}{i\omega} \left[ \chi(Q_h,\omega) \frac{\partial n}{\partial \mu} - 1 \right],$$

and the spin relaxation is mapped to the ordinary density diffusion.

In cold atomic gases condition $Q_h \ll \langle k \rangle$ is not anymore valid, and $Q_h > \langle k \rangle$ regime is easily achievable. This leads to new physical effects, which we will consider when discussing the spin dynamics of cold fermions.

4.1. Disordered electrons

Now we apply Eq. (30) to a noninteracting disordered two-dimensional electron gas with a momentum relaxation time $\tau$ and study possible regimes of spin dynamics. In the semiclassical regime, where $k_F \ell \gg 1$, (with $k_F$ being the Fermi momentum and $\ell$ being the electron free path) corresponding to...
the summation of ladder diagrams, one obtains:
\[
\mathcal{D}(Q_h, \omega) = \frac{\mathcal{K}(Q_h, \omega)}{1 - \mathcal{K}(Q_h, \omega)}, \tag{31}
\]
\[
\mathcal{K}(Q_h, \omega) = \frac{1}{2\pi} \int \frac{d\theta}{1 - i\omega\tau + i\Omega_{so}\cos\theta}. \tag{32}
\]

Here we introduce notation \(\Omega_{so} \equiv Q_h v_F\) for the maximum spin precession rate, which is the only parameter dependent on the SO coupling in this approach with \(\Omega_{so}\tau = k_F Q_h\).

In terms of the dual transformed system the clean limit \(\Omega_{so}\tau \gg 1\) corresponds to a reversible, purely ballistic washing out of the helix texture, where electrons freely move between points of different spin polarization, thus eventually removing the nonuniform spin polarization, as shown in Fig. 4.1, right panel. Here limit \(\mathcal{D}(Q_h, \omega) \approx \mathcal{K}(Q_h, \omega)\), and integration in Eqs. (28) and (32) yields:
\[
S_\perp(t) = S_\perp(0) J_0(\Omega_{so} t), \tag{33}
\]

where \(J_0(\Omega_{so} t)\) is the Bessel function, that is the time scale of the spin vanishing is of the order of \(L_{so}/v_F\). On the other hand, the momentum-dependent precession rate \(\Omega_{so}(\mathbf{k}) = \Omega_{so}\cos\phi\), where \(\phi\) is the angle between \(\mathbf{k}\) and \(\nu\) yields:
\[
S_\perp(t) = S_\perp(0) \int \cos(\Omega_{so} t \cos\phi) \frac{d\phi}{2\pi} = S_\perp(0) J_0(\Omega_{so} t), \tag{34}
\]
same as in Eq.(33).

An opposite \(\Omega_{so}\tau \ll 1\) limit is a well-defined diffusion (Fig. 4.1, left panel), where the Dyakonov-Perel’ mechanism for spin relaxation has to be
Fig. 3. Left panel: weak magnetic field. Right panel: strong magnetic field.

restored. Indeed, in this case the main contribution to the diffusion Green's function is given by:

\[ \mathcal{D}(Q_h, \omega) = \frac{1}{DQ_h^2 - i\omega}, \tag{35} \]

where \( D = v_F^2 \tau / 2 \) is the diffusion coefficient. By integrating Eq. (28) with \( \mathcal{D}(Q_h, \omega) \) of Eq. (35) we immediately obtain the Dyakonov-Perel’ mechanism with the spin relaxation rate \( \Gamma_s = DQ_h^2 \):

\[ S_\perp(t) = S_\perp(0) \exp(-DQ_h^2 t), \tag{36} \]

Moreover, the factor 1/2 in the definition of \( D \) acquires an interesting physical meaning in terms of spin precession: it corresponds to the angular averaging of the precession rate \( \langle \Omega_{so}(k) \rangle = \Omega_{so}^2 / 2 \).

An intermediate regime of \( \Omega_{so} \tau \sim 1 \), can be investigated numerically and shows a crossover between the oscillating Bessel function-like behavior to the exponential Dyakonov-Perel’ decay. At short times \( \Omega_{so} \tau \ll 1 \), the behavior of spin is universal: \( S_\perp(t) = S_\perp(0) (1 - \Omega_{so}^2 t^2 / 2) \) due to the unperturbed precession of the spins. Analysis of Eqs. (28), (31), and (32) shows that \( \Omega_{so} \tau = 1 \) is a singular point. With the decrease in \( \Omega_{so} \tau \) in the domain \( \Omega_{so} \tau > 1 \), the first node in \( S_\perp(t) = 0 \) shifts to a larger time, and the negative value regions become more shallow. At \( \Omega_{so} \tau < 1 \), spin \( S_\perp(t) \) is always positive.

We proceed with the spin relaxation in magnetic field where taking into account spin-orbit coupling would be difficult otherwise. We begin with the clean limit \( \omega_c \tau \gg 1 \) with the cyclotron frequency \( \omega_c = eB/mc \). In this case the motion of the electrons is very close to circular and the diffusion kernel can be presented in the form:

\[ \mathcal{D}(\rho' - \rho, t) = \frac{1}{2\pi d(t)} \delta [||\rho' - \rho| - d(t)|], \]
where the electron displacement: \( d(t) = 2R_c |\sin(\omega_c t/2)| \), where \( R_c = v_F/\omega_c \) is the cyclotron radius. Integration in Eq.(25) immediately yields:

\[
S_\perp(t) = S_\perp J_0 \left( \frac{\Omega_0}{\omega_c} \sin \left( \frac{\omega_c t}{2} \right) \right).
\]

In the weak field limit as shown in Fig. 4.1 (left panel) here understood as \( R_c \gg L_\omega \), the result coincides with Eq.(33), as expected. In the opposite case of \( \omega_c \gg \Omega_0 \), no relaxation occurs. In terms of precession it can be understood as very fast changes in the direction of the spin-orbit field at the frequency \( \omega_c \) keeping the total spin out of the relaxation. In terms of the relaxation of the nonuniform density, this implies that only electrons at the distances less than \( 2R_c \) can achieve the given point. If \( R_c Q_h = \Omega_0/\omega_c \) is much less than one, the density in the point of interest experiences only subtle variation with time of the order of \( (R_c Q_h)^2 \) since in the achievable region the density is close to the density in its center (see Fig. 4.1, right panel).

In the regime where \( \omega_c \tau \gg 1 \) is not yet very large, the electron mobility decreases due to the Lorenz force preventing electron propagation as \( (1 + \ell^2/R_c^2)^{-1} \) compared to the \( B = 0 \) case. By Einstein relation, the diffusion coefficient is renormalized by the factor \( D / (1 + \ell^2/R_c^2) \), too. As a result, spin relaxation time \( \tau_s(B) = 1/\Gamma_s \) increases as \( \tau_s(B) = \tau_s(0) (1 + \ell^2/R_c^2) \). This slowing down of the spin relaxation was first found by solving the kinetic equation for the density matrix and confirmed by a complicated quantum mechanical calculation later.

4.2. Interacting cold Fermions and spin drag

Cold fermions bring new physics in the problem of spin dynamics for at least two reasons. First reason is the relative strength of spin-orbit coupling, which can be much stronger than in semiconductors. Second reason is the absence of disorder and the role of interatomic interaction which can be important here.

For definiteness we consider the configuration of fields generated in the experiments of Ref. 14. This corresponds to \( \hbar = (0, 0, 1), \ Q = (Q_h, 0, 0), \) and \( \B = (B, 0, 0) \), which translates to the following helicoidal structure of the transformed Zeeman field

\[
\tilde{\B}(r) = B [\hat{x} \cos(Q_h x) + \hat{y} \sin(Q_h x)],
\]

which forms the same spin density configuration. As for two-dimensional electrons, here we consider a response to a week \( B(t) = B\theta(-t) \) field.
Since the inter-particle interaction and the trap potential are spin independent the form of $\hat{H}_{\text{int}}$ is unchanged by the SU(2) rotation. Therefore the physical system with the pure gauge SOC subjected to a uniform polarizing field $\mathbf{B} = \hat{z}B$ is mapped to a dual system without SOC in the presence of an inhomogeneous Zeeman field - the magnetic helix of Eq. (37). For a harmonic trap with $U(r) = m\Omega^2 r^2/2$ the density depends on $r$ as $n(r) = n(0) (1 - r^2/R^2)^{3/2}$, where the radius $R = \sqrt{2\mu/m\Omega^2}$. Here $\mu$ is Fermi energy at $r = 0$. In the experiment, the parameter $k_F(0)R \sim \mu/\Omega\tau$, which determines the applicability of the local approximation is larger than 100. Below we assume $Q_h R \gg 1$, so that the trap hosts many helix periods.

To obtain the susceptibility which allows one to explore the behavior in the full range of SOC, densities, and temperatures, we employ a spin version of the “conserving” relaxation time scheme by Mermin. The dynamical spin response function, which recovers the correct static and high-frequency response, and respects the local spin conservation becomes

$$\chi_{\sigma\sigma}(Q_h, \omega) = \frac{\chi_0(Q_h, \omega + i\mathcal{F}_s)}{1 - \frac{i\mathcal{F}_s}{\omega + i\mathcal{F}_s} \left[ 1 - \frac{\chi_0(Q_h, \omega + i\mathcal{F}_s)}{\chi_0(Q_h, 0)} \right]},$$  \hspace{1cm} (38)

where $\mathcal{F}_s$ is the corresponding relaxation rate. Here $\chi_0(q, \tilde{\omega})$ at a complex $\tilde{\omega}$ is the Lindhard function

$$\chi_0(q, \tilde{\omega}) = \sum_p \frac{f_{p+q/2} - f_{p-q/2}}{\tilde{\omega} - (\epsilon_{p+q/2} - \epsilon_{p-q/2})}$$  \hspace{1cm} (39)

with $f_p$ being the Fermi function, and $\epsilon_p = p^2/2m$.

The spin drag relaxation rate $\Gamma_s$ enters the Mermin’s scheme as a phenomenological parameter originating from interatomic collisions. Note that for repulsive interactions our consideration is valid at any temperature, while for the attraction the system has to be in the normal phase. Although collisions conserve the total momentum, they cause a friction between different spin species, and thus lead to relaxation and damped response in the spin channel. Strictly speaking, the spin drag rate $\mathcal{F}_s$ depends on $Q_h$. It is however clear that for large $q \gtrsim k_F$ the average spin relaxes mostly because of fast nonuniform precession (ballistic motion). The spin drag contribution becomes qualitatively important in the opposite limit of $Q_h \ll k_F$. Therefore, assuming a weak short-range interaction characterized by a $s$-wave scattering length $a_s$, we adopt the conventional expression based
on the Born approximation at $q = 0$

$$F_s = \left(\frac{4\pi a_s}{m}\right)^2 \sum_k \frac{k^2}{3mnT} \int_0^\infty \frac{d\omega}{\pi} \frac{\sinh^2(\omega/2T)}{[\text{Im}\chi_0(k,\omega)]^2},$$  

(40)

where $n$ is the concentration of atoms. The low-$T$ behavior $F_s \sim E_F (k_F a_s)^2 (T/E_F)^2$ is typical for the Fermi liquid. In the high-$T$ limit $F_s \sim E_F (k_F a_s)^2 (T/E_F)^{1/2}$ describes the frequency of collisions between particles with a scattering cross-section $\sim a_s^2$ moving at the thermal velocity $v_T \sim \sqrt{T/m}$. For the repulsive interaction the Stoner instability criterion for formation of a ferromagnetic state yields $43 k_F a_s < \pi/2$, and, therefore $F_s \leq E_F (T/E_F)^2$. As a result, irreversible spin relaxation in in the degenerate gas is slow.

The scheme (38)-(40) captures all main physical effects which influence the spin relaxation. The difference in the Fermi functions in Eq. (39) describes the modification of the initial state by the SO coupling. It selects a part of the momentum space with the particles polarized by the initial Zeeman field, as it is shown in Fig. ?? for the very strong coupling. The difference in energies in the denominator in Eq. (39) determines the frequency of the spin precession, while the imaginary shift of $\omega$ by $iF_s$ describes the spin drag effect. Now we use Eqs. (38)-(40) to explore different regimes of spin dynamics and discuss those not achievable in the solid-state experiments.

We consider first the degenerate gas, $T \ll E_F$. If SOC is weak, so that $Q_h \ll k_F$, only atoms near the Fermi surface participate in the dynamics. The characteristic time scale is the time for ballistically traversing the period of the spin helix $L_{so}/v_F$. As we have seen in the example of disordered electron gas, the relation of this time to the spin drag rate $F_s$ determines the spin dynamics. If $v_F F_s^{-1}/L_{so} \gg 1$, the dynamics is ballistic and the evolution of the total spin is dominated by the momentum-dependent spin precession of the Fermi-surface particles. Apparently this is always the case at sufficiently low temperatures where $F_s \sim T^2$. In the opposite limit of $v_F F_s^{-1}/L_{so} \ll 1$ the dynamics is diffusive yielding the conventional Dyakonov-Perel’ behavior. Therefore, the spin drag rate can be obtained from the relaxation of the uniform spin density without generating an inhomogeneous spin distribution in the real space. With the increase in the temperature the ballistic regime crosses over to the diffusive one at $T_{db} \sim (v_F/a_s)\sqrt{Q_h/k_F}$. Condition $T_{db} \ll E_F$ implies $Q_h \ll k_F (k_F a_s)^2$ for the SOC strength. Taking into account that, e.g., in Ref. 14, $k_F a_s \sim 10^{-2}$, we conclude that to reach the diffusive regime one has to increase the in-
Apart from determining the spin drag rate, the temperature modifies the dynamics via broadening the Fermi distribution in Eq. (39). Indeed, the broadening of the velocity distribution by \( \delta v_F \sim (T/E_F) v_F \) causes the broadening in the precession rate by the typical value of \( \delta \Omega = (T/E_F) q v_F \).

In the ballistic regime the resulting time dependence of the total spin is

\[
S(t) = S(0) \frac{\sin(v_F q t)}{\sinh(\pi q t \delta \Omega)} \frac{\pi T}{E_F},
\]

where the rapid increase in \( \sinh(\pi q t \delta \Omega) \) at \( t > 1/\delta \Omega \) leads to the exponential damping in the spin oscillations initially following the \( \sin(v_F Q h t)/v_F Q h t \) time dependence. At \( \delta \Omega/F_s \ll 1 \), that is at \( T \gg T_{db}/E_F \) the asymptotic behavior is determined by \( F_s \) rather than by \( \delta \Omega \).

For strong SOC the dynamics is mostly ballistic, but still quite nontrivial and diverse. Pronounced large amplitude oscillations appear for ultrastrong SOC with \( Q_h \gg k_F \) since we have two well separated spin-split Fermi-spheres. The position of the sphere center determines the mean precession rate, and the broadening is determined by the Fermi momentum \( k_F \). An interesting boundary regime occurs at \( Q_h = 2k_F \) when the spin-split Fermi-surfaces touch each other and the time dependence of the spin takes the form

\[
S(t) = S(0) \frac{\sin^2(4E_F t)}{(4E_F t)^2}.
\]

Now we consider the high-temperature case, \( T \gg E_F \). In this limit the mean free path \( \sim v_T/F_s \) does not depend on the temperature. Therefore the dynamical regime is temperature-independent – the dynamics is ballistic if \( (Q_h/k_F)(k_F a_s)^2 \gg 1 \) and diffusive otherwise. The condition of diffusive relaxation is equivalent to the condition \( T_{db} \ll E_F \). As a result, if \( Q_h \geq k_F (k_F a_s)^2 \), the spin dynamics is ballistic at any temperature. In
this case the width of the thermal Maxwell distribution is much larger than
the scale of SOC and the inhomogeneous precession is equivalent to the
Fourier transform of the Gaussian momentum distribution with the coordi-
inate $q/m$. For a weak SOC in the ballistic regime the spin relaxation
is position-independent Gaussian $S(t) = S(0) \exp \left[ - (v_T Q h t)^2 / 6 \right]$ with the
time scale of $1/Q h v_T$. At larger scattering length $a_s$, that is at larger $F_s$,
we enter the diffusive regime where the Gaussian damping slows down to
the exponential Dyakonov-Perel’ relaxation with the timescale $F_s / Q^2 h v_T^2$.
In the nondegenerate gas the function $S(t)$ is practically always monotonic.
To detect signatures of the oscillatory spin precession one needs to go to
the ultrastrong SOC and to satisfy experimentally the condition $Q h \gg k_T$.

It is worth noting that in general for the weak SOC the spin dynamics
in the ballistic limit is determined by the parameter $v_{at} Q h$, where $v_{at}$ is
the typical atomic velocity. Since this is the only characteristics of the
low-energy excitation spectrum entering Eq. (39). For the strong coupling
with clearly separated spin-split momentum distributions we have two main
parameters describing spin dynamics: spin precession rate of the order of
$Q^2 h / 2m$ and spin relaxation rate of the order of $Q h v_{at}$.

5. Conclusions.

We have shown how to formulate the theory of macroscopic spin dynamics
in an interacting system of fermions where the spin-orbit coupling can be
described as a pure gauge, and, therefore, removed by a local SU(2) rotation
in the spin subspace. This allows one to map the spin dynamics on the
generalized density diffusion and to present the entire spin dynamics in
terms of momentum and frequency-dependent spin susceptibility for a non-
magnetic system without a spin-orbit coupling. This includes the effects of
the orbital motion in a magnetic field on the spin dynamics as well. The
momentum entering the susceptibility is, essentially, the inverse period of
the spin helix produced by the spin-orbit coupling, being the only parameter
dependent on that coupling. In cold Fermi gases the spin dynamics can be
mapped on the spectrum of particle-hole excitations, while the spin
relaxation is due to the spin drag effect.

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